Ninth Annual Upper Peninsula
High School Math Challenge

Northern Michigan University (Marquette, MI, USA)
Saturday April 14, 2018

Individual Problems — Solutions

Problem 1

Dan is negotiating to buy a used car. The listed price is \$4000. Dan offers 20% less than the listed price. The owner counters with a price 20% greater than Dan's offer. What is the owner's counteroffer?

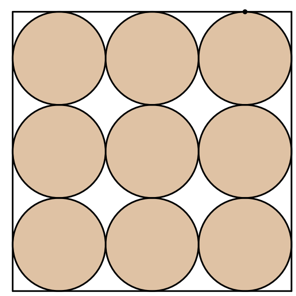
Answer: \$3840

Dan offers: \$4000 x 0.80 = \$3200
Counteroffer: \$3200 x 1.20 = \$3840

Problem 2

A 3 ft. × 3 ft. dartboard is designed with nine congruent circles that are mutually tangent to each other and to the square. Provided that the dart hits the board, what is the probability that it lands in the non-shaded region?

(Express your answer in terms of π and/or radicals, if appropriate. Do not approximate as a decimal.)



Answer: $1-\frac{\pi}{4}$

$$\text{Area shaded} = \pi\left(\frac{1}{2}\right)^2 \cdot 9 = \frac{9}{4}\pi \text{ ft}^2$$

$$\text{Nonshaded area} = 9 - \frac{9\pi}{4} \text{ ft}^2$$

$$\text{P (not shaded)} = \frac{9-\frac{9\pi}{4}}{9} = 1-\frac{\pi}{4}$$

Problem 3

What is the last digit of the number 7^{89}?

Answer: 7

	Cycle
$7^1 = 7$	(1)
$7^2 = 49$	(2)
$7^3 = 343$	(3)
$7^4 = 2401$	(4)
$7^5 = 16{,}807$	(1)
$7^6 = 117{,}649$	(2)

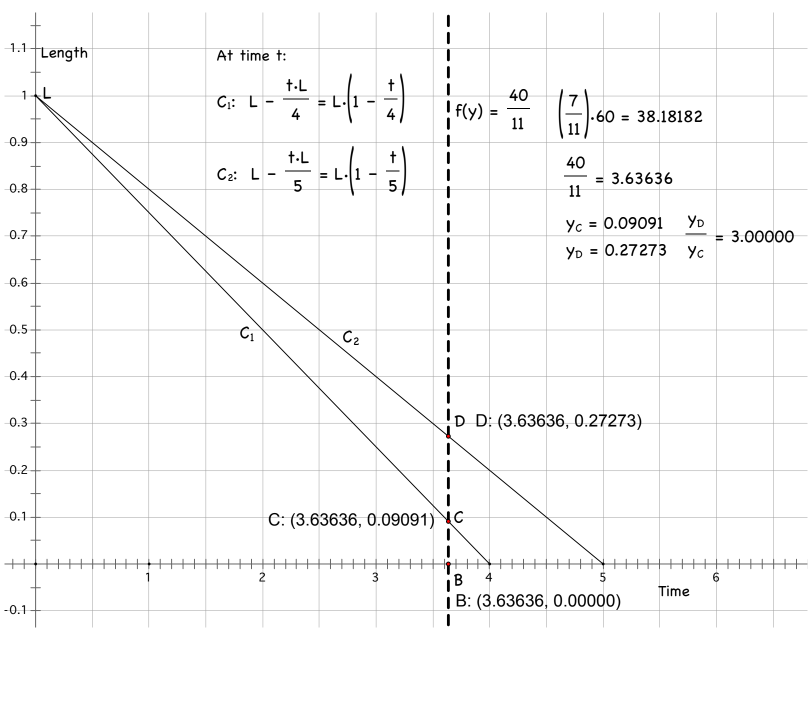
units digits cycle: 7,9,3,1,7,9,3,1,...

$89_{\text{mod } 4} = 1 \Rightarrow$ units digit of $7^{89} = 7$

Problem 4

Two candles have the same length. One is consumed uniformly in four hours, the other in five hours. If they are lit at the same time, how long will it take until one candle is three times as long as the other?

Answer: $\frac{40}{11}$ $\left(\text{or } 3\frac{7}{11}\right)$ **hours or 3 hours, $38.\overline{18}$ minutes**



$$L\cdot\left(1 - \frac{t}{5}\right) = 3\cdot L\cdot\left(1 - \frac{t}{4}\right)$$

$$1 - \frac{t}{5} = 3 - \frac{3t}{4}$$

$$\frac{3t}{4} - \frac{t}{5} = 2$$

$$\frac{15t - 4t}{20} = \frac{11t}{20} = 2$$

$$11t = 40$$

$$t = \frac{40}{11} \text{ hours}$$

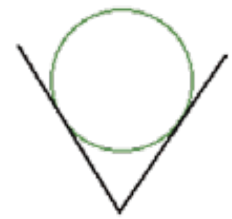
Problem 5

The Richter scale is a base-10 logarithmic measure of the amplitude of an earthquake. How many times stronger is an earthquake that measures 6.9 compared to one that measures 5.4?

Answer: $\sqrt{1000} = 10\sqrt{10}$

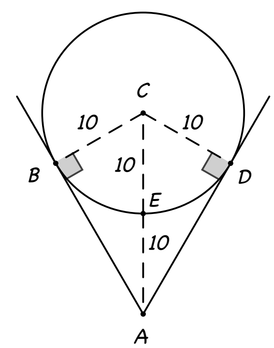
$$\frac{10^{6.9}}{10^{5.4}} = 10^{1.5} = 10^{3/2} = \sqrt{10^3} = \sqrt{1000} = 10\sqrt{10} \quad \text{(approx. 31.6)}$$

Problem 6



A ball of radius 10 cm is dropped into a V-shaped gutter. A vertical cross section containing the ball's center is shown. If the shortest distance from the surface of the ball to the vertex of the gutter is 10 cm, what is the angle formed by the sides of the gutter?

Answer: 60°



CB = CD = CE = 10 cm (radii)
EA = 10 cm (shortest distance to vertex A)
$\angle CBA = \angle CDA = 90°$ (ball is tangent to gutter)
$\therefore$ CA = 20 cm $\Rightarrow$ 30° — 60° — 90° triangles
$\therefore \angle CAB = \angle CAD = 30°$
$\therefore \angle BAD = 60°$

Problem 7

A stack of 100 nickels is 6.25 inches high. To the nearest cent, how much would a stack of nickels 8 feet high be worth?

Answer: $76.80

8 feet = 96 inches. Dividing 96 inches by 6.25 yields 15.36.
100 nickels are worth $5.00 and there are 15.36 groups of 100 nickels in the stack
$\therefore$ the stack's value is $5.00 × 15.36 = $76.80.

Problem 8

Determine the largest prime divisor of 87! + 88!.

Answer: 89

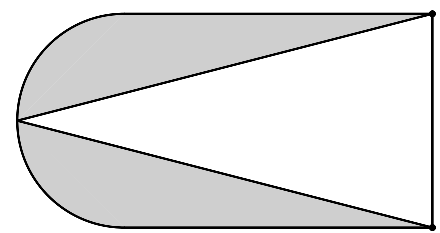
$87! + 88! = 87! + (87! \times 88) = 87! \times (1 + 88) = 87! \times 89$

All the primes dividing 87! are less than 89, which is prime, so 89 is the largest prime divisor of the product.

Problem 9

A semicircle is added to the shorter side of a rectangle having the dimensions 23 inches by 16 inches, and an isosceles triangle is inscribed as shown. Find the area of the shaded region.

(Express your answer in terms of π and/or radicals, if appropriate. Do not approximate as a decimal.)



Answer: $120 + 32\pi$ square inches

Area of the rectangle: $23 \times 16 = 368 \left(in^2\right)$

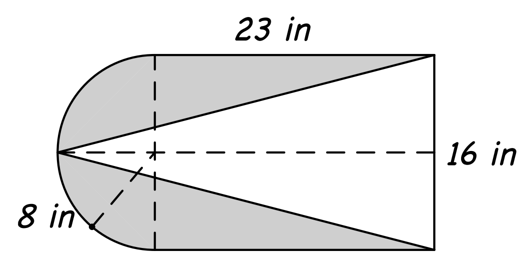
Radius of semicircle: 8 in

Area of semicircle: $\dfrac{\pi \cdot 8^2}{2} = \dfrac{64\pi}{2} = 32\pi \left(in^2\right)$

Total area: $368 + 32\pi \left(in^2\right)$

Area of triangle: $\dfrac{16 \cdot (23 + 8)}{2} = 8 \cdot 31 = 248 \left(in^2\right)$

Area shaded $= (368 + 32\pi) - 248 = 120 + 32\pi \left(in^2\right)$



Problem 10

The numbers one through seven are drawn from a hat without replacement. What is the probability that all the odd numbers are chosen first?

Answer: $\dfrac{1}{35}$

1,2,3,4,5,6,7 $\rightarrow$ 4 odd, 3 even

Permutations of 7 numbers with 4 odd, 3 even is

$$\frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = 7 \cdot 5 = 35$$

Only one of those arrangements is O,O,O,O,E,E,E

$\therefore$ P (first 4 odd) $= \dfrac{1}{35}$